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Doubly Periodic Solution for Nonlinear Schrödinger's Equation With Higher Order Polynomial Law Nonlinearity

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Abstract This paper obtains the travelling wave solutions of the nonlinear Schrödinger's equation with higher order polynomial law nonlinearity. The doubly periodic wave solution of this equation is obtained. The numerical simulation is also included.

Keywords Nonlinear Schrödinger's equation · Travelling waves · Elliptic functions · Integrals of motion

1 Introduction

The nonlinear Schrödinger's equation (NLSE) plays a vital role in various areas of physical, biological and engineering sciences. It appears in many applied fields like Fluid Dynamics, Nonlinear Optics, Plasma Physics and Protein Chemistry. The NLSE that is going to be studied in this paper is given by [1–12].

$$iq_t + \frac{1}{2}q_{xx} + F\big(|q|^2\big)q = 0. \tag{1}$$

In (1), F is a real-valued algebraic function where $F(|q|^2)q : C \mapsto C$. Considering the complex plane C as a two-dimensional linear space R^2, it can be said that the function $F(|q|^2)q$ is k times continuously differentiable so that one can write

$$F\big(|q|^2\big)q \in \bigcup_{m,n=1}^{\infty} C^k\big((-n,n)\times(-m,m); R^2\big). \tag{2}$$

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In (1), q is the dependent variable while x and t are the independent variables that represents space and time respectively. The first term in (1) represents the time evolution term while the second term is due to the group velocity dispersion and the third term accounts for nonlinearity. This is a nonlinear partial differential equation that is not integrable, in general. The nonintegrability is not necessarily related to the nonlinear term in (1). Higher order dispersion, for example, can also make the system non-integrable while it still remains Hamiltonian.

Equation (1) governs the evolution of a wave packet in a weakly nonlinear and dispersive medium and has thus arisen in diverse fields such as water waves, plasma physics, nonlinear optics. One other application of this equation is in pattern formation, where it has been used to model some nonequilibrium pattern forming systems. In particular, this equation is now widely used in the optics field as a good model for optical pulse propagation in nonlinear fibers. It is also known to support soliton solutions for particular forms of the nonlinearity in F. In this paper, however, the interest is going to be on obtaining the doubly periodic solution to (1) by the travelling wave ansatz.

1.1 Integrals of Motion

An important property of nonlinear evolution equations is that it has conserved quantities also known as *Integrals of Motion*. In fact, (1) has three integrals of motion. They are the energy (E), linear momentum (M) and Hamiltonian (H) which are respectively given by

$$E = \int_{-\infty}^{\infty} |q|^2 dx, \tag{3}$$

$$M = i \int_{-\infty}^{\infty} (q^* q_x - q q_x^*) dx \tag{4}$$

and

$$H = \int_{-\infty}^{\infty} \left[\frac{1}{2} |q_x|^2 - f(I) \right] dx \tag{5}$$

where

$$f(I) = \int_0^I F(\xi) d\xi \tag{6}$$

with the intensity $I = |q|^2$. The first conserved quantity is also known as the *mass, wave action* or *plasmon number*; in optics, however, it is called the *wave power* while mathematically, it is known as the L_2 norm. The Hamiltonian is one of the most fundamental notions in mechanics and more generally in the theory of conservative dynamical systems with finite or even infinite degrees of freedom. The most useful approach in the soliton theory of conservative non-integrable Hamiltonian system is a representation on the plane of conserved quantities namely the Hamiltonian-versus-energy diagrams [4]. One can see that (1) can now be written in a canonical form as

$$i q_t = \frac{\delta H}{\delta q^*}, \tag{7}$$

$$i q_t^* = -\frac{\delta H}{\delta q} \tag{8}$$

where in (7) and (8) the right sides denote the Fréchet derivative.

1.2 Travelling Waves

A travelling wave is a solution of the NLSE that represents a wave of permanent form which does not change its shape during propagation and moves with a constant speed. The wave may be localized or periodic. In order to seek a travelling wave solution of the NLSE that is given by (1), introduce the ansatz

$$q(x,t) = Ag[B(x-\bar{x}(t))]e^{i(-\kappa x+\omega t+\sigma_0)} \tag{9}$$

where the function g represents the shape of the wave form described by (1) and it depends on the type of nonlinearity in it. Also, in (9) A and B respectively represent the amplitude and width of the wave, κ is the wave frequency while ω is the wave number, σ_0 is the phase while $\bar{x}$ gives the mean position, so that the velocity of the soliton is given by

$$v = \frac{d\bar{x}}{dt}. \tag{10}$$

The soliton width and the amplitude are related as $B = \lambda(A)$ where the functional form λ depends on the type of nonlinearity in (1). Thus, from (9), one easily gets

$$q_t = -ABvg'(\tau)e^{i\phi} - i\omega Ag(\tau)e^{i\phi} \tag{11}$$

where, the phase ϕ is

$$\phi(x,t) = -\kappa x + \omega t + \sigma_o \tag{12}$$

and

$$\tau = B(x-\bar{x}). \tag{13}$$

Also, from (9) one can get

$$q_{xx} = AB^2 g''(\tau)e^{i\phi} - 2i\kappa ABg'(\tau)e^{i\phi} - \kappa^2 Ag(\tau)e^{i\phi}. \tag{14}$$

Substituting (11) and (14) into (1) simplifies it to

$$-iBvg' + \omega g + 1/2B^2 g'' - i\kappa Bg' - 1/2\kappa^2 g + gF\left(A^2g^2\right) = 0. \tag{15}$$

From (15), equating the real and imaginary parts yields

$$ABg'(\kappa + v) = 0 \tag{16}$$

and

$$B^2 g'' - \left(\kappa^2 - 2\omega\right)g + 2gF\left(A^2g^2\right) = 0 \tag{17}$$

so that from (16), one obtains

$$\kappa = -v. \tag{18}$$

Now, multiplying both sides of (17) by g', then integrating and choosing the integration constant to be zero since the wave profile is such that q, q_x and q_{xx} approach zero as $|x| \to \infty$ gives

$$B^2(g')^2 - (\kappa^2 - 2\omega)g^2 + 2\int (g^2)' F(A^2 g^2) dg = 0. \tag{19}$$

On separation of variables and integrating once more, (19) leads to

$$x - vt = \int \frac{dg}{[(\kappa^2 - 2\omega)g^2 - 2\int (g^2)' F(A^2 g^2) dg]^{\frac{1}{2}}}. \tag{20}$$

Equation (20) can be integrated if the law of nonlinearity is known. This integral will be evaluated for the case of higher order polynomial law nonlinearity as will be seen in the following section.

2 Higher Order Polynomial Law

In this paper, the type of nonlinearity that is going to be studied is given by higher order polynomial law nonlinearity that is given by

$$F(s) = s + \nu_1 s^2 + \nu_2 s^3 \tag{21}$$

so that the NLSE that is being studied in this paper takes the form

$$iq_t + \frac{1}{2}q_{xx} + (|q|^2 + \nu_1 |q|^4 + \nu_2 |q|^6)q = 0. \tag{22}$$

For this law of nolinearity, although the energy and the linear momentum stays the same, the Hamiltonian takes the form

$$H = \int_{-\infty}^{\infty} \left(\frac{1}{2}|q_x|^2 - \frac{1}{2}|q|^4 - \frac{\nu_1}{3}|q|^6 - \frac{\nu_2}{4}|q|^8 \right) dx. \tag{23}$$

It is to be noted that the case where $\nu_2 = 0$, with $\nu_1 \neq 0$ is known as the parabolic law nonlinearity. This case is already studied exhaustively in the context of nonlinear fiber optics [5].

For this kind of nonlinearity, (20) reduces to

$$x - vt = \int \frac{B\sqrt{6} dg}{g\sqrt{6(\kappa^2 - 2\omega) - 6A^2 g^2 - 4\nu_1 A^4 g^4 - 3\nu_2 A^6 g^6}} \tag{24}$$

which integrates to

$$\begin{aligned} &-\frac{x - vt}{\sqrt{6}Bg}\sqrt{g^2(12\kappa^2 - 6A^2 g^2 - 4\nu_1 A^4 g^4 - 3\nu_2 A^5 g^6 - 24\omega) g_3} \\ &= \Pi\left[1 - \frac{g_2}{g_3}, \sin^{-1}\sqrt{\frac{g^2 - g_3}{g_2 - g_3}}, \frac{g_2 - g_3}{g_1 + g_3}\right]\sqrt{\frac{g^2 - g_1}{g_3 - g_1}} \\ &\times \sqrt{\frac{(g^2 - g_2)(g^2 - g_3)}{(g_2 - g_3)^2}}(g_3 - g_2) \end{aligned} \tag{25}$$

where Π is the incomplete elliptic integral of the third kind that is defined as

$$\Pi(n;\phi,\alpha)=\int_0^{\phi}\frac{d\theta}{(1-n\sin^2\theta)\sqrt{1-\sin^2\alpha\sin^2\theta}} \tag{26}$$

and

$$g_1=\frac{1}{9\nu_2A^2}\left[-4\nu_1+\frac{2(8\nu_1^2-27\nu_2)}{R^{\frac{1}{3}}}+R^{\frac{1}{3}}\right], \tag{27}$$

$$g_2=\frac{1}{18\nu_2A^2}\left[-8\nu_1+\frac{2i(i+\sqrt{3})(8\nu_1^2-27\nu_2)}{R^{\frac{1}{3}}}+i\left(i+\sqrt{3}\right)R^{\frac{1}{3}}\right], \tag{28}$$

$$g_3=\frac{1}{36\nu_2A^6}\left[-16\nu_1A^4+\frac{4(-2)^{\frac{2}{3}}(8\nu_1^2-27\nu_2)}{R^{\frac{1}{3}}}-2^{\frac{4}{3}}\left(1+i\sqrt{3}\right)R^{\frac{1}{3}}\right] \tag{29}$$

with R being given by

$$\begin{aligned}R&=-64\nu_1^3+324\nu_1\nu_2+729\nu_2^2\kappa^2-1458\nu_2^2\omega\\&\quad+\sqrt{8\left(-8\nu_1^2+27\nu_2\right)^3-64\nu_1^3+324\nu_1\nu_2+729\nu_2^2(\kappa^2-2\omega)^2}.\end{aligned} \tag{30}$$

This solution was plotted numerically and is shown in Fig. 1. The solution represents the wave profile $|q(x,t)|^2$ for x between -0.01 and 0.01 while t varies from 0 to 1.

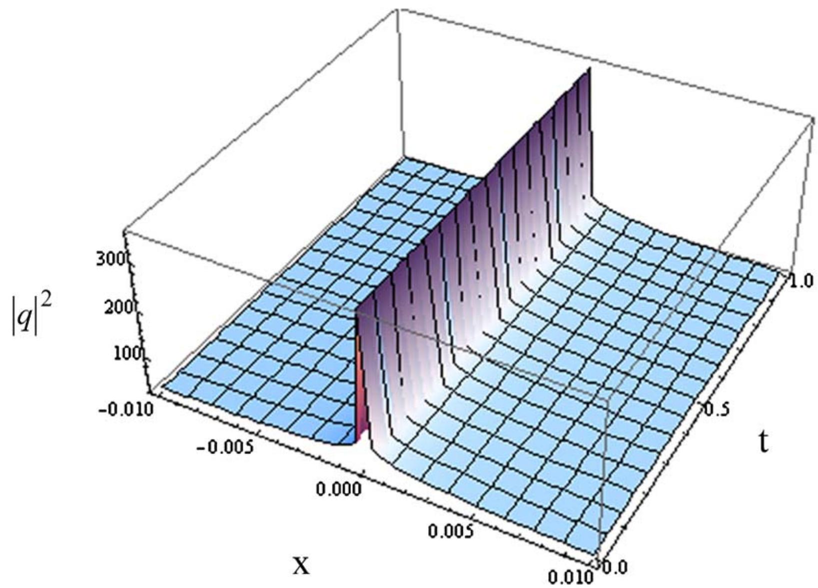

Fig. 1 Numerical simulation of the solution

3 Conclusions

In this paper, the travelling wave solution to the NLSE with higher order polynomial law nonlinearity is studied. The doubly periodic wave solution is obtained and that result is plotted numerically. In future, this result will be extended to the case of triple power law nonlinearity and those results will be published elsewhere.

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